

# A GENERALIZATION OF WOLSTENHOLME'S HARMONIC SERIES CONGRUENCE

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ABSTRACT. Let  $A, B$  be two non-zero integers. Define the Lucas sequences  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$  by

$$u_0 = 0, \quad u_1 = 1, \quad u_n = Au_{n-1} - Bu_{n-2} \text{ for } n \geq 2$$

and

$$v_0 = 2, \quad v_1 = A, \quad v_n = Av_{n-1} - Bv_{n-2} \text{ for } n \geq 2.$$

For any  $n \in \mathbb{Z}^+$ , let  $w_n$  be the largest divisor of  $u_n$  prime to  $u_1, u_2, \dots, u_{n-1}$ . We prove that for any  $n \geq 5$

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2 - 1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{w_n^2},$$

where  $\Delta = A^2 - 4B$ .

## 1. INTRODUCTION

Let  $A, B$  be two non-zero integers. Define the Lucas sequence  $\{u_n\}_{n=0}^{\infty}$  by

$$u_0 = 0, \quad u_1 = 1 \text{ and } u_n = Au_{n-1} - Bu_{n-2} \text{ for } n \geq 2.$$

Also its companion sequence  $\{v_n\}_{n=0}^{\infty}$  is given by

$$v_0 = 2, \quad v_1 = A \text{ and } v_n = Av_{n-1} - Bv_{n-2} \text{ for } n \geq 2.$$

Let  $\Delta = A^2 - 4B$  be the discriminant of  $\{u_n\}_{n=0}^{\infty}$  and  $\{v_n\}_{n=0}^{\infty}$ . It is easy to show that

$$v_n = \alpha^n + \beta^n$$

and

$$u_n = \sum_{j=0}^{n-1} \alpha^j \beta^{n-1-j} = \begin{cases} n\alpha^{n-1} & \text{if } \Delta = 0, \\ (\alpha^n - \beta^n)/(\alpha - \beta) & \text{otherwise,} \end{cases}$$

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where

$$\alpha = \frac{1}{2}(A + \sqrt{\Delta}), \beta = \frac{1}{2}(A - \sqrt{\Delta}).$$

Let  $p \geq 5$  be a prime. The well-known Wolstenholme's harmonic series congruence asserts that

$$\sum_{j=1}^{p-1} \frac{1}{j} \equiv 0 \pmod{p^2}. \quad (1.1)$$

In [3], Kimball and Webb proved a generalization of (1.1) involving the Lucas sequences. Let  $r$  be the rank of apparition of  $p$  in the sequence  $\{u_n\}_{n=0}^{\infty}$ , i.e.,  $r$  be the least positive integer such that  $p \mid u_r$ . Kimball and Webb showed that

$$\sum_{j=1}^{r-1} \frac{v_j}{u_j} \equiv 0 \pmod{p^2} \quad (1.2)$$

provided that  $\Delta = 0$  or  $r = p \pm 1$ .

In this paper we will extend the result of Kimball and Webb to arbitrary Lucas sequences. For any positive integer  $n$ , let  $w_n$  be the largest divisor of  $u_n$  prime to  $u_1, u_2, \dots, u_{n-1}$ . Here  $w_n$  was firstly introduced by Hu and Sun [2] in an extension of the Lucas congruence for Lucas'  $u$ -nomial coefficients.

**Theorem 1.1.** *Let  $n \geq 5$  be a positive integer. Then*

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2 - 1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{w_n^2}. \quad (1.3)$$

It is easy to check that either all  $u_n$  are odd when  $n \geq 1$ , or one of  $u_2 = A$  and  $u_3 = A^2 - B$  is even. So  $w_n$  is odd for any  $n > 3$ . Also we can verify that either  $u_n$  is prime to 3 for each  $n \geq 1$ , or 3 divides one of  $u_2, u_3$  and  $u_4 = A^3 - 2AB$ . Hence  $3 \nmid w_n$  provided that  $n > 4$ . Finally we mention that  $w_n$  is always prime to  $v_n$  when  $n \geq 3$ . Indeed, since

$$u_n = Au_{n-1} - Bu_{n-2} \text{ and } (w_n, Au_{n-1}) = (w_n, u_2u_{n-1}) = 1,$$

we have  $w_n$  and  $B$  are co-prime. And from

$$v_n = u_{n+1} - Bu_{n-1} = Au_n - 2Bu_{n-1},$$

it follows that  $(w_n, v_n) = (w_n, 2Bu_{n-1}) = 1$ .

The Fibonacci numbers  $F_0, F_1, \dots$  are given by

$$F_0 = 0, F_1 = 1 \text{ and } F_n = F_{n-1} + F_{n-2} \text{ for } n \geq 2.$$

And the Lucas numbers  $L_0, L_1, \dots$  are given by

$$L_0 = 2, L_1 = 1 \text{ and } L_n = L_{n-1} + L_{n-2} \text{ for } n \geq 2.$$

Then by Theorem 1.1, we immediately have

**Corollary 1.2.** *Let  $p \geq 5$  be a prime. Let  $n$  be the least positive integer such that  $p \mid F_n$ . Then we have*

$$\sum_{j=1}^{n-1} \frac{L_j}{F_j} \equiv \frac{5(n^2 - 1)}{6} \cdot \frac{F_n}{L_n} \pmod{p^2}. \quad (1.4)$$

The proof of Theorem 1.1 will be given in the next section.

## 2. PROOF OF THEOREM 1.1

For any  $n \in \mathbb{N}$ , the  $q$ -integer  $[n]_q$  is given by

$$[n]_q = \frac{1 - q^n}{1 - q} = 1 + q + \cdots + q^{n-1}.$$

Now we consider  $[n]_q$  as the polynomial in the variable  $q$ . Recently Shi and Pan [5] established a  $q$ -analogue of (1.1) for prime  $p \geq 5$ :

$$\sum_{j=1}^{p-1} \frac{1}{[j]_q} \equiv \frac{p-1}{2}(1-q) + \frac{p^2-1}{24}(1-q)^2[p]_q \pmod{[p]_q^2}. \quad (2.1)$$

Let

$$\Phi_n(q) = \prod_{\substack{1 \leq k < n \\ (k, n) = 1}} (q - \zeta_n^k)$$

be the  $n$ -th cyclotomic polynomial, where  $\zeta_n = e^{2\pi i/n}$ . We know that  $\Phi_n(q)$  is a polynomial with integral coefficients, and clearly  $\Phi_n(q)$  is prime to  $[j]_q$  for each  $1 \leq j < n$ . Indeed, using similar method, we can easily extend (2.1) as follows:

**Proposition 2.1.** *Let  $n$  be a positive integer. Then*

$$24 \sum_{j=1}^{n-1} \frac{1}{[j]_q} \equiv 12(n-1)(1-q) + (n^2-1)(1-q)^2[n]_q \pmod{\Phi_n(q)^2}. \quad (2.2)$$

For the proof of (2.1) and (2.2), the reader may refer to [5]. From (2.2), we deduce that

$$\begin{aligned} 12 \sum_{j=1}^{n-1} \frac{1+q^j}{[j]_q} &= 12 \sum_{j=1}^{n-1} \frac{2 - (1-q^j)}{[j]_q} \\ &= 24 \sum_{j=1}^{n-1} \frac{1}{[j]_q} - 12(n-1)(1-q) \\ &\equiv (n^2-1)(1-q)^2[n]_q \pmod{\Phi_n(q)^2}. \end{aligned}$$

And the above congruence can be rewritten as

$$\left(12 \sum_{j=1}^{n-1} \frac{1+q^j}{[j]_q} - (n^2-1)(1-q)(1-q^n)\right) \prod_{j=1}^{n-1} [j]_q \equiv 0 \pmod{\Phi_n(q)^2}.$$

Since  $\Phi_n(q)$  is a primitive polynomial, by Gauss's lemma (cf. [4], Chapter IV Theorem 2.1 and Corollary 2.2), there exists a polynomial  $G(q)$  with integral coefficients such that

$$\left(12 \sum_{j=1}^{n-1} \frac{1+q^j}{[j]_q} - (n^2-1)(1-q)(1-q^n)\right) \prod_{j=1}^{n-1} [j]_q = G(q)\Phi_n(q)^2. \quad (2.3)$$

*Proof of Theorem 1.1.* When  $\Delta = 0$ , the theorem reduces to Wolstenholme's congruence (1.1). So below we assume that  $\Delta \neq 0$ , i.e.,  $\alpha \neq \beta$ . Let  $p$  be a prime with  $p \mid w_n$ , and let  $m$  be the integer such that  $p^m \mid w_n$  but  $p^{m+1} \nmid w_n$ . Obviously we only need to show that

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{p^{2m}}$$

for each such  $p$  and  $m$ .

Let  $\mathbb{K} = \mathbb{Q}(\sqrt{\Delta})$  and let  $\mathcal{O}_{\mathbb{K}}$  be the ring of algebraic integers in  $\mathbb{K}$ . Clearly  $\alpha, \beta \in \mathcal{O}_{\mathbb{K}}$ . Let  $(p)$  denote the ideal generated by  $p$  in  $\mathcal{O}_{\mathbb{K}}$ . We know that if  $\left(\frac{\Delta}{p}\right) = -1$  then  $(p)$  is prime in  $\mathcal{O}_{\mathbb{K}}$ , where  $\left(\frac{\cdot}{p}\right)$  is the Legendre symbol. Also there exist two distinct prime ideal  $\mathfrak{p}$  and  $\mathfrak{p}'$  such that  $(p) = \mathfrak{p}\mathfrak{p}'$  provided that  $\left(\frac{\Delta}{p}\right) = 1$ . Finally when  $p \mid \Delta$ ,  $(p)$  is the square of a prime ideal  $\mathfrak{p}$ . The reader can find the details in [1]. Let

$$\mathfrak{P} = \begin{cases} (p) & \text{if } \left(\frac{\Delta}{p}\right) = -1 \text{ or } 0, \\ \mathfrak{p} & \text{if } \left(\frac{\Delta}{p}\right) = 1. \end{cases}$$

Obviously either  $\alpha$  or  $\beta$  is prime to  $\mathfrak{P}$ , otherwise we must have  $\mathfrak{P}$  is not prime to  $u_j$  for any  $j \geq 2$ , which implies that  $p \mid u_j$ . Without loss of generality, we may assume that  $\beta$  is prime to  $\mathfrak{P}$ .

**Lemma 2.2.** *Let  $p$  be a prime and  $k \in \mathbb{Z}$ . Suppose that  $\left(\frac{\Delta}{p}\right) = 1$  and  $(p) = \mathfrak{p}\mathfrak{p}'$ . Then for any  $m \in \mathbb{Z}^+$ ,  $\mathfrak{p}^m \mid k$  implies that  $p^m \mid k$ .*

*Proof.* Observe that  $\sigma : \sqrt{\Delta} \mapsto -\sqrt{\Delta}$  is an automorphism over  $\mathbb{K}$ . Also we know that  $\sigma(\mathfrak{p}) = \mathfrak{p}'$ . Hence

$$\mathfrak{p}'^m = \sigma(\mathfrak{p}^m) \mid \sigma(k) = k.$$

Since  $\mathfrak{p}$  and  $\mathfrak{p}'$  are distinct prime ideals, by the unique factorization theorem, we have  $(p)^m = \mathfrak{p}^m \mathfrak{p}'^m$  divides  $k$ .  $\square$

Now it suffices to prove that

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} \equiv \frac{(n^2-1)\Delta}{6} \cdot \frac{u_n}{v_n} \pmod{\mathfrak{P}^{2m}}.$$

For any  $l \in \mathbb{Z}^+$ , let

$$\Phi_l(\alpha, \beta) = \beta^{\varphi(l)} \Phi_l(\alpha/\beta) = \prod_{\substack{1 \leq d \leq l \\ (d,l)=1}} (\alpha - \zeta_l^d \beta),$$

where  $\varphi$  is the Euler totient function. Apparently  $\Phi_l(\alpha, \beta) \in \mathcal{O}_{\mathbb{K}}$ . Notice that

$$u_l = \frac{\alpha^l - \beta^l}{\alpha - \beta} = \prod_{1 < d, d|l} \beta^{\varphi(d)} \Phi_d(\alpha/\beta) = \prod_{1 < d, d|l} \Phi_d(\alpha, \beta).$$

Hence  $u_l$  is always divisible by  $\Phi_l(\alpha, \beta)$ . Then we have  $w_n$  divides

$$\Phi_n(\alpha, \beta) = \frac{u_n}{\prod_{\substack{1 < d < n \\ d|n}} \Phi_d(\alpha, \beta)}$$

since  $w_n$  is prime to  $u_d$  whenever  $1 \leq d < n$ .

Substituting  $\alpha/\beta$  for  $q$  in (2.3), and noting that

$$u_j = \beta^{j-1} \frac{1 - (\alpha/\beta)^j}{1 - \alpha/\beta} = \beta^{j-1} [j]_{\alpha/\beta} \text{ and } v_j = \beta^j (1 + (\alpha/\beta)^j),$$

we obtain that

$$\left( 12\beta^{-1} \sum_{j=1}^{n-1} \frac{v_j}{u_j} - (n^2-1)\beta^{-n-1}(\alpha-\beta)^2 u_n \right) \prod_{j=1}^{n-1} \beta^{1-j} u_j = \beta^{-2\varphi(n)} G(\alpha/\beta) \Phi_d(\alpha, \beta)^2.$$

As  $(w_n, 6) = 1$  and  $\mathfrak{P}$  is prime to  $\beta$ , we conclude that

$$\sum_{j=1}^{n-1} \frac{v_j}{u_j} - \frac{(n^2-1)\Delta}{12} \beta^{-n} u_n \equiv 0 \pmod{(\mathfrak{P}^m)^2}.$$

Finally, since

$$\alpha^n = \frac{1}{2}(v_n + u_n \sqrt{\Delta}) \text{ and } \beta^n = \frac{1}{2}(v_n - u_n \sqrt{\Delta}),$$

we have

$$\alpha^n \equiv \beta^n \equiv v_n/2 \pmod{w_n}.$$

All are done.  $\square$

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## REFERENCES

- [1] E. Hecke, *Lectures on the Theory of Algebraic Numbers* (Graduated texts in mathematics; 77), Springer-Verlag, New York, 1981.
- [2] H. Hu and Z.-W. Sun, *An extension of Lucas' theorem*, Proc. Amer. Math. Soc. **129** (2001), 3471–3478.
- [3] W. A. Kimball and W. A. Webb, *Some generalizations of Wolstenholme's theorem*, Applications of Fibonacci numbers, Vol. 8, Kluwer Acad. Publ., Dordrecht, 1999, 213–218.
- [4] S. Lang, *Algebra*. Revised third edition. (Graduated texts in mathematics; 211), Springer-Verlag, New York, 2002.
- [5] L.-L. Shi and H. Pan, *A  $q$ -analogue of Wolstenholme's harmonic series congruence*, Amer. Math. Monthly, to appear, arXiv:math.NT/0507495.

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